NORMAL DERIVATIVE FOR BOUNDED DOMAINS WITH GENERAL BOUNDARY

GONG GUANGLU, QIAN MINPING AND MARTIN L. SILVERSTEIN

ABSTRACT. Let D be a general bounded domain in the Euclidean space R^n. A Brownian motion which enters from and returns to the boundary symmetrically is used to define the normal derivative as a functional for f with f, ∇f and Δf all in L^2 on D. The corresponding Neumann condition (normal derivative $= 0$) is an honest boundary condition for the L^2 generator of reflected Brownian notion on D. A conditioning argument shows that for D and f sufficiently smooth this general definition of the normal derivative agrees with the usual one.

Introduction. B_t, $t \geq 0$, is standard Brownian motion on the Euclidean space R^n. The case when the dimension $n = 1$ is well understood and so we assume always that $n \geq 2$.

D is a bounded open connected subset of R^n. An important role will be played by

$$\sigma = \inf\{t > 0 \colon B_t \text{ is not in } D\},$$

the *hitting time for* the complement $R^n \backslash D$.

The *absorbed process* X_t, $t \geq 0$, is

$$X_t = B_t \quad \text{for } 0 \leq t < \sigma, \qquad X_t = \partial \quad \text{for } t \geq \sigma$$

with ∂ a *dead point* adjoined to D as an isolated point. All functions are *understood to vanish at* ∂.

The absorbed transition operators P_t^0, $t \geq 0$, with action

$$P_t^0 f(x) = E_x f(X_t)$$

form a strongly continuous symmetric semigroup on L^2, the Hilbert space of real valued functions on D that are square integrable with respect to Lebesgue measure dx.

The (strong) *generator* A^0 is selfadjoint and negative definite. By the spectral theorem as formulated and proved in Chapter IX of Segal and Kunze (**1968**), the operator $(-A^0)^{1/2}$ is a well-defined positive definite selfadjoint operator. The *absorbed Dirichlet space* is the pair (F^0, E) where

$$(0.1) \qquad\qquad F^0 = \mathrm{domain}(\{-A^0\}^{1/2}),$$

$$E(f, g) = \int dx (-A^0)^{1/2} f(x) (-A^0)^{1/2} g(x), \qquad f, g \text{ in } F^0.$$

CONVENTION 1. All integrals are understood to be over D unless otherwise specified.

We can explicitly identify (F^0, E). The space F^0 is the set of functions f in L^2 such that if f is extended to all of R^n by setting $f(x) = 0$ for x in the complement $R^n \backslash D$ then ∇f is in L^2 on R^n.

CONVENTION 2. The Laplacian Δ, the gradient ∇ and all partial derivatives are understood in the sense of distributions. If no set is specified then the distribution theory is on D.

CONVENTION 3. "L^p on K" means the space of real valued p-integrable functions on K. If no K is specified then $K = D$ is understood.

CONVENTION 4. ∇f in L^2 means that its norm $|\nabla f|$ is.

The form E is

$$(0.2) \qquad\qquad E(f, g) = \frac{1}{2} \int dx\, \nabla f(x) \cdot \nabla g(x)$$

with "$\cdot$" denoting the usual inner product between two n-dimensional vectors. Also we will use the auxiliary form E_a defined for $a > 0$ by

$$E_a(f, g) = E(f, g) + a \int f(x) g(x)\, dx.$$

Reflecting Brownian motion on D is easily specified in terms of its Dirichlet space (F^r, E). The space F^r is the set of functions f in L^2 such that also ∇f is in L^2 (no condition on an extension of f). The form E is still (0.2).

REMARK. If D has a smooth boundary (Lipschitz is enough) then Theorem 7.3(iii) in SMP implies that F^0 is the set of f in F^r which vanish modulo polar sets on the boundary of D.

NOTATION. The abbreviation SMP is used in place of the reference Silverstein (**1974**).

CONVENTION 5. Functions f in F^r are represented by their quasi-continuous versions as specified in Theorem 3.13 in SMP. In particular f is well defined modulo polar sets.

The strong L^2 generator A^r of the reflecting transition semigroup P_t^r, $t \geq 0$, is also selfadjoint and nonpositive definite. From general principles f in L^2 belongs to domain (A^r) and $A^r f = g$ if and only if the following two conditions are satisfied.

GEN1. f is in F^r.

GEN2. For all h in F^r

$$(0.3) \qquad\qquad E(f, h) + \int dx\, g(x) h(x) = 0.$$

We will see in §4 that (0.3) can be checked separately for h in $C_{\text{com}}^\infty(D)$ and for h bounded and harmonic and in F^r.

CONVENTION 6. "h is harmonic" means on D unless another set is specified. "h in $C_{\text{com}}^\infty(K)$" means h is C^∞ and has compact support contained in K.

For h in $C_{\text{com}}^\infty(D)$, integration by parts shows that (0.3) is equivalent to

$$(0.4) \qquad\qquad \tfrac{1}{2} \Delta f(x) = g(x).$$

Suppose now that h is harmonic and belongs to F^r. If both f and h are sufficiently smooth on the closure of D and if D itself has a sufficiently smooth boundary

then the classical "divergence theorem" can be applied to give

(0.5)
$$\frac{1}{2}\int dx\,\nabla f(x)\cdot\nabla h(x)+\int dx\,g(x)h(x)$$
$$=\frac{1}{2}\int dx(\nabla f(x)\cdot\nabla h(x)+\Delta f(x)h(x))$$
$$=\frac{1}{2}\int dx\,\mathrm{div}(h\nabla f(x))$$
$$=\frac{1}{2}\int_{\partial D} d\sigma(z)h(z)D_n f(z)$$

with $D_n f(z)$ denoting the "classical" exterior normal derivative. This suggests that the L^2 reflected generator is determined by the classical Neumann boundary condition

(0.6)
$$D_n f(z)=0.$$

In fact the authors are not sure that (0.6) works as a boundary condition even for D smooth. Instead we use a functional version of the normal derivative introduced in Silverstein (**1976**) and prove that this works for general bounded D (no smoothness assumed).

To discuss the functional normal derivative $\partial f/\partial n(\varphi)$ we must introduce some machinery. In §2 we describe an "approximate Brownian motion" which enters from and returns to the boundary with symmetric distribution. The process is denoted X_t, $\sigma^* < t < \sigma$, with σ^*,σ the times of entrance from and return to the boundary. The sample space and sample space measure are denoted Ω_∞ and P_∞, with expectation functional denoted E_∞. An abstract boundary $\partial^\# D$ is introduced in §5 which represents all harmonic h in F^r and such that X_σ and X_{σ^*} are defined almost everywhere as points in $\partial^\# D$ by passage to the limit in t. A function φ defined on $\partial^\# D$ is a test function if it represents, or equivalently is the restriction to $\partial^\# D$ of, a harmonic function in F^r. The *normal derivative* $\partial f/\partial n$ is the functional defined on test functions φ by

(0.7) $$\partial f/\partial n(\varphi)=\operatorname*{Lim}_{k\uparrow\infty} E_\infty\varphi(X_{\sigma^*})\{f(X_{\sigma^*})-f(X_{\sigma(D_k)})\}I(\sigma(D_k)<\infty).$$

with D_k, $k\geq 1$, an increasing sequence of open sets with closure contained in D and with union D. The symbol $\sigma(D_k)$ represents the first hitting time for D_k, with the condition "$\sigma(D_k)<\infty$" meaning that D_k is hit before return to the boundary.

Our main result is the following theorem which will be proved in §6.

THEOREM. *Assume that f is in F^r and that Δf is in L^2. Then $\partial f/\partial n(\varphi)$ exists for bounded test functions φ and if h is the harmonic function represented by φ*

(0.8) $$\partial f/\partial n(\varphi)=\frac{1}{2}\int\{\nabla f(x)\cdot\nabla h(x)+\Delta f(x)h(x)\}.$$

In particular f belongs to domain (A^r) if and only if

(0.9) $$\partial f/\partial n(\varphi)=0$$

for all bounded test functions φ.

We will see in §7 that for D smooth we can arrange matters so that $\partial^{\#} D$ is the Euclidean boundary ∂D. Then (0.8) and (0.5) together imply that for D and f smooth

$$(0.10) \qquad \partial f/\partial n(\varphi) = \frac{1}{2} \int \varphi(z) D_n f(z) \, d\sigma(z)$$

with $d\sigma$ denoting the usual surface measure on ∂D. We believe that it is instructive to verify (0.10) directly from (0.7) and so in §8 we treat two special cases, the interval $(0, 1)$ and the half plane R_+^2, even though they do not fit the hypotheses of the main text. In both cases we can explicitly compute the joint distribution of X_{σ^*} and $X_{\sigma(D_k)}$, assuming a particular choice for the D_k. In §7 we use a more robust technique based on a conditioning argument suggested to one of the authors by Richard F. Gundy. This works for general smooth and bounded D.

We emphasize that this functional normal derivative and its connection with boundary conditions is the new result. The reflected Brownian motion itself was constructed by Fukushima (**1967**). Fukushima constructs the process using the Martin-Kuramochi boundary. We introduce a different boundary $\partial^{\#} D$ in §5 and do not construct the process—not needed for our result.

For K a subset of D its *hitting time* is

$$(0.11) \qquad \sigma(K) = \inf\{t > 0 : X_t \text{ is in } K\}$$

with the usual understanding that $\sigma(K) = +\infty$ when the set on the right is empty. The *hitting operator* H^K is defined by

$$H^K f(x) = E_x I(\sigma(K) < \infty) f(X_{\sigma(K)}).$$

REMARK. In SMP a set N is said to be *polar* if it has capacity 0 in the sense of Definition 3.3 there. By Proposition 4.5 there, this implies

$$(0.12) \qquad P_x(\sigma(N) < \infty) = 0$$

for almost every x. In general this is weaker than the standard concept which requires (0.12) for all x. Fortunatley, for Brownian motion, the two concepts are equivalent. This and much more follows from the very interesting results in Fukushima (**1971**). So we can use the term "polar" without worrying about which concept is intended.

1. The absorbed space. The absorbed transition operators can be represented

$$(1.1) \qquad P_t^0 f(x) = \int P_t^0(x, y) f(y) \, dy$$

with $P_t^0(x, y)$ jointly continuous and symmetric:

$$(1.2) \qquad P_t^0(x, y) = P_t^0(y, x), \qquad x, y \text{ in } D.$$

The potential density

$$G^0(x, y) = \int_0^\infty dt \, P_t^0(x, y)$$

and for $a > 0$ there resolvent density

$$G_a^0(x, y) = \int_0^\infty dt \, e^{-at} P_t^0(x, y)$$

are finite and jointly continuous for $x \neq y$. Also $G^0(x,y)$ is harmonic in each variable for $x \neq y$ and $G_a^0(x,y)$ is in the same way a-harmonic—that is, annihilated by $\frac{1}{2}\Delta - a$. Of course (1.2) implies the same symmetry for $G^0(x,y)$ and $G_a^0(x,y)$. The operators G^0 and G_a^0 are defined as P_t^0 is in (1.1). All this can be found in §4 of Port and Stone (**1978**).

For K a subset of D

$$(1.3) \qquad\qquad h^K(x) = H^K 1(x).$$

If K has closure contained in D then by Chapter 6, §5, ibid., there exists a Radon measure μ_K on the boundary of K such that

$$(1.4) \qquad\qquad h_K(x) = \int G^0(x,y)\, d\mu_K(y).$$

In general μ_K and h_K are referred to as the *equilibrium measure and potential* for K.

The first two lemmas are known. We include our own proofs as a matter of convenience.

LEMMA 1.1. *The absorbed semigroup* P_t^0, $t \geq 0$, *is strongly continuous on* L^2.

PROOF. It is enough to show that for a dense set of f in L^2, in particular for bounded f

$$(1.5) \qquad\qquad P_t^0 f \to f \text{ strongly in } L^2$$

as $t \downarrow 0$. We take as known that the lemma is true for the transition operators

$$P_t f(x) = E_x f(B_t)$$

for the original Brownian motion on R^n. Therefore it is enough to show that the difference $E_x I(\sigma \leq t) f(B_t) \to 0$ strongly in L^2. But this is clear since $P_x(\sigma > 0) = 1$ for x in D and since D has bounded measure.   Q.E.D.

By §14 in Silverstein (**1976**) the Dirichlet space F for Brownian motion on R^n is the set of functions f such that both f and ∇f are in L^2 on R^n. The Dirichlet form is (0.2) with the integral over R^n instead of D. Then Theorem 7.3 in SMP justifies the description of the absorbed space (F^0, E) given in the Introduction.

By §1 in SMP every Dirichlet space (F, E) is *contractive*. This means that if $g(t)$, a real valued function of real t, satisfies

$$(1.6) \qquad\qquad g(0) = 0, \quad |g(s) - g(t)| \leq |s - t|$$

and if f is in F, then also $g(f)$ is in F and

$$(1.7) \qquad\qquad E(g(f), g(f)) \leq E(f,f).$$

Important examples for $g(t)$ are $t^+, t^-, |t|, a \wedge t, a \vee t$ and $\frac{1}{2}(1 \wedge t^2)$.

The absorbed Dirichlet space (F^0, E) satisfies the following two conditions.

REG1. $F^0 \cap C_{\text{com}}^\infty(D) = C_{\text{com}}^\infty(D)$.

REG2. $F^0 \cap C_{\text{com}}^\infty(D)$ is E_1 dense in F^0.

To verify Reg2 it is enough to approximate f in F^0 by convolutions $f * \varphi_k$ with $\varphi_k(x) = k^n \varphi(kx)$ where φ is in $C_{\text{com}}^\infty(k)$ and integrates to 1, after first approximating by functions in F^0 which are bounded and have compact support in D.

For the former use bounded contractions. For the latter subtract projections onto complements of open sets with compact closure in D. (See Theorem 7.3 in SMP.)

Reg1 and Reg2 together with the elementary fact that dx charges every nonempty open subset of D imply regularity in the sense of §2 in SMP.

Nothing is gained by introducing the extended Dirichlet space $F^0_{(e)}$ of §1 in SMP since our assumption that D is bounded implies that $F^0_{(e)} = F^0$. This follows from the next lemma.

LEMMA 1.2. *The absorbed Dirichlet space F^0 is complete with respect to the Dirichlet norm E.*

PROOF. Consider first the case when D is the cube

$$Q = \{x \text{ in } R^n : 0 < x_j < R \text{ for } 1 \le j \le n\}.$$

Then an eigenbasis for the absorbed generator A^0 is the set of functions

$$\varphi_k(x) = \prod_{j=1}^{n} \sin\{\pi k_j x_j / R\}$$

with $k = (k_1, \ldots, k_n)$ an n-tuple of positive integers. The eigenvalue for φ_k is $-\sum_{j=1}^{n}(k_j \pi / R)^2$ and this implies that for any f in F^0

$$\int dx \, f^2(x) \le (R^2/n\pi^2) E(f, f)$$

and completeness with respect to E follows from the known completeness of F^0 with respect to E_1. In general D is contained in such a cube Q and completeness with respect to E follows since the absorbed space for D is a subspace of the one for Q. Q.E.D.

2. An approximate Brownian motion. Up to now we have been concerned only with Brownian motion starting in the interior of D and eventually reaching the boundary. For this we have implicitly used the *standard sample space* Ω. This is the set of trajectories ω with $\omega(t)$ defined for $t \ge 0$, taking values in $D \cup \{\partial\}$ and satisfying the following two conditions.

TRAJ1. There exists a time $\sigma > 0$ such that $\omega(t) = \partial$ for $t \ge \sigma$ and $\omega(t)$ is in D for $0 \le t < \sigma$.

TRAJ2. $\omega(t)$ is continuous for $t \ne \sigma$.

The coordinate variables X_t are defined on ω in Ω by $X_t(\omega) = \omega(t)$. The σ-algebra F_s on Ω is the one generated by the X_t. The sample space probabilites P_x, x in D are defined on F_s.

In this section we introduce a process which enters D from the boundary and then acts like absorbed Brownian motion. An appropriate sample space is the *extended sample space* Ω_∞. This is the set of trajectories ω with $\omega(t)$ defined for all real t, taking values in $D \cup \{\partial\}$ and satisfying:

TRAJ1'. There exist times $\sigma^* < \sigma$ such that $\omega(t) = \partial$ for $t \le \sigma^*$ and for $t \ge \sigma$ and such that $\omega(t)$ is in D for $\sigma^* < t < \sigma$.

TRAJ2'. $\omega(t)$ is continuous for $t \ne \sigma$ or σ^*.

On Ω_∞ the coordinate variables X_t are defined as on Ω and the σ-algebra generated by them is denoted F_∞.

The hitting times σ_K are defined on Ω_∞ as on Ω:

$$\sigma_K = \inf\{t > \sigma^* : X_t \text{ is in } K\}$$

and $+\infty$ if the set on the right is empty. The shift θ_K maps Ω_∞ into Ω according to the formula

$$\theta_K \omega(t) = \omega(\sigma_K + t)$$

when $\sigma_K < \infty$.

The following theorem is proved in §5 in SMP.

THEOREM 2.1. *There exists a unique measure P_∞ on F_∞ with the following properties.*

1. *For Borel K with closure contained in D and for $\phi \geq 0$ and F_s measurable on Ω, the composition $\phi \cdot \theta_K$ is F_∞ measurable on Ω_∞. If $\psi \geq 0$ is defined on Ω_∞ and measurable with respect to the σ-algebra generated by $X_{t \wedge \sigma(K)}$, t real (the past) and if $\psi = 0$ when $\sigma_K = \infty$, then*

$$(2.1) \qquad E_\infty(\psi)(\phi \cdot \theta_K) = E_\infty \psi E_{X(\sigma\{K\})} \phi.$$

Also

$$(2.2) \qquad E_\infty I(\sigma(K) < \infty)\phi \cdot \theta_K = \int d\mu_K(x) E_x \phi$$

with $d\mu_K$ equilibrium measure for K.

2. *For $f \geq 0$ and Borel measurable on D*

$$(2.3) \qquad E_\infty \int_{\sigma^*}^{\sigma} f(X_t)\,dt = \int f(x)\,dx.$$

3. *Let the "time reversal operator" ρ be defined on ω in Ω_∞ so that $\rho\omega(\sigma^* + t) = \omega(\sigma - t)$. Then P_∞ is ρ invariant (time reversal invariant). That is, for $\phi \geq 0$ and F_∞ measurable on Ω_∞, also $\phi \cdot \rho$ is and*

$$(2.4) \qquad E_\infty \phi \cdot \rho = E_\infty \phi.$$

REMARK. "$\phi \cdot \theta_K$ is measurable" in Conclusion 1 and "$\phi \cdot \rho$ is measurable" in Conclusion 3 mean measurable with respect to the P_∞ completion of F_∞.

The adjective "approximate" refers to X_t, $T \geq \sigma^*$, not being Brownian motion in general. The terminology as well as the idea go back to G. A. Hunt (**1960**) who worked in a discrete time context. A continuous time version was first constructed by M. Weil (**1970**).

It is not hard to see that P_∞ is an unconditioned version of the excursion law as defined for example in Burdzy (**1985**).

3. The reflected Dirichlet space. Our candidate for the reflected Dirichlet space is the pair (F^r, E) described in the Introduction. To show that (F^r, E) really is a Dirichlet space on L^2 as in Fukushima (**1969**) and as in SMP, we must establish contractivity as described in §1 and we must show that for $a > 0$ the function space F^r is complete relative to

$$(3.1) \qquad E_a(f, g) = E(f, g) + a \int dx\, f(x)g(x), \qquad f, g \text{ in } F^r,$$

already introduced in the introduction.

LEMMA 3.1. F^r is E_a complete, any $a > 0$.

PROOF. Let f_n, $n \geq 1$, in F^r be E_a Cauchy. Then there exists scalar f and vector valued g in L^2 such that $\nabla f_n \to g$ and $f_n \to f$ in L^2. The lemma will be proved when we show that $\nabla f = g$. But this follows since for vector valued φ in $C_{\text{com}}^\infty(D)$

$$
\begin{aligned}
\int dx \, g(x) \cdot \varphi(x) &= \lim_{n \uparrow \infty} \int dx \, \nabla f_n(x) \cdot \varphi(x) \\
&= -\lim_{n \uparrow \infty} \int dx \, f_n(x) \nabla \cdot \varphi(x) \\
&= -\int dx \, f(x) \nabla \cdot \varphi(x) \\
&= dx \, \nabla f(x) \cdot \varphi(x). \quad \text{Q.E.D.}
\end{aligned}
$$

LEMMA 3.2. If f is in F^r and if g satisfies (1.6), then also $g(f)$ is in F^r and

$$(3.2) \qquad E(g\{f\}, g\{f\}) \leq E(f, f).$$

PROOF. Let K be an open set with closure contained in D. It suffices to show that for all such K, the distribution $\nabla g(f)$ is in L^2 on K and

$$(3.3) \qquad \int_K |\nabla g(f)|^2(x) \, dx \leq \int_K |\nabla f|^2(x) \, dx.$$

If g and f are both in C^1, then by the classical chain rule $\nabla g(f) = g'(f) \nabla f$ which implies for all x in K

$$(3.4) \qquad |\nabla g(f)|(x) \leq |\nabla f|(x)$$

and this certainly implies (3.3). Suppose now that f is in C^1 but g is not. There exists an at most countably infinite set S of times t such that $\int_K dx \, I(f(x) = t) > 0$. Assume at first that 0 is not in S. Then there exist

$$(3.5) \qquad t_{i,j}, \qquad i \geq 1, \ -\infty < j < +\infty,$$

such that none of the $t_{i,j}$ belong to S and such that

$$(3.6) \qquad t_{i,0} = 0, \quad t_{i,j} < t_{i,j+1} < t_{i,j} + 1/i,$$
$$\inf_j t_{i,j} = -\infty, \quad \sup_j t_{i,j} = +\infty.$$

For each i let g_i be the piecewise linear function which interpolates g at the points $t_{i,j}$, $-\infty < j < +\infty$. Then g_i satisfies (1.6), the chain rule $\nabla g_i(f) = g_i'(f) \nabla f$ is valid almost everywhere on K and therefore so is (3.4). Also $g_i(f) \to g(f)$ in L^2 and by the argument on p. 1.12 in SMP (Banach and Saks) Cesàro sums of a subsequence of the $\nabla g_i(f)$ converge in L^2 to a vector valued function h. By the proof of Lemma 3.1, $h = \nabla g(f)$ and we are done with this case.

If 0 is in S choose a_k, $k \geq 1$, not in S such that $a_k \to 0$ and let $f_k = f - a_k$. The above argument is valid for f_k and so $g(f_k)$ is in F^r with smaller E norm than f_k. Certainly $g(f_k) \to g(f)$ in L^2 and again we finish with the "argument of Banach and Saks."

For f not in C^1 use convolutions to find f_k, $k \geq 1$, in C^1 on A such that f_k and ∇f_k converge in L^2 on K to f and ∇f respectively. For a subsequence $f_k \to f$ and therefore $g(f_k) \to g(f)$ almost everywhere on K. By the previous result

$$\sup_k \int_K dx \, |\nabla g(f_k)|^2(x) \leq \sup_k \int_K dx \, |\nabla f_k|^2(x)$$

which is finite since $\nabla f_k \to \nabla f$ in L^2 on K. One more application of the "argument of Banach and Saks" finishes the proof.   Q.E.D.

Thus (F^r, E) is indeed a Dirichlet space on L^2. By Fukushima (**1969**) or by SMP there exists a strongly continuous symmetric semigroup of submarkovian (actually markovian) transition operators P_t^r with strong L^2 generator A^r, selfadjoint and nonpositive definite, such that (0.1) is true with F^0, A^0 replaced by F^r, A^r.

4. The Dirichlet principle. For $a > 0$ let H_a denote the E_a orthogonal complement in F^r of F^0.

CONVENTION. We use the same symbol H_a to denote the operator which implements E_a orthogonal projection onto the space H_a.

LEMMA 4.1. *A function h in F^r belongs to the subspace H_a if and only if h (has a version which) is C^∞ on D and satisfies*

(4.1) $$\tfrac{1}{2}\Delta h = ah.$$

PROOF. If h in F^r is C^∞ and satisfies (4.1) then for any φ in $C_{\text{com}}^\infty(D)$

(4.2) $$E_a(h, \varphi) = \frac{1}{2} \int dx \, \nabla h(x) \nabla \varphi(x) + a \int dx \, h(x)\varphi(x)$$

$$= \int dx \, \varphi(x)\{-\tfrac{1}{2}\Delta h(x) + ah(x)\}$$

$$= 0.$$

Such φ are dense in F^0 by Reg2 in §1 and so h is in H_a.

Conversely, assume that h is in H_a and let B be an open ball with closure contained in D. The function h coincides on the closure of B with a function h_0 in F^0. (Multiply h by φ in $C_{\text{com}}^\infty(D)$ with $\varphi = 1$ everywhere on B.) The absorbed space for B is contained in F^0 and so $E_a(h_0, g) = E_a(h, g) = 0$ with g in the absorbed space for B. (Replacement of h_0 by h is allowed by the local nature of E.) By Theorem 7.3-(i) in SMP

(4.3) $$h_0(x) = E_x e^{-a\tau} h_0(X_\tau), \qquad x \text{ in } B,$$

with τ the first exit time from B. (That is $\tau = \sigma_{D\backslash B}$.) Of course h_0 can be replaced by h in (4.3) and so

(4.4) $$h(x) = E_x e^{-a\tau} h(X_\tau)$$

$$= E_x \left\{ 1 - a \int_0^\tau dt \, e^{-at} \right\} h(X_\tau)$$

$$= E_x h(X_\tau) - \int_B dy \, a G_a^B(x, y) E_y h(X_\tau)$$

$$= E_x h(X_\tau) - a \int_B dy \, G^B(1 - aG_a^B)(x, y) E_y h(X_t)$$

with $G^B(x,y)$, $G_a^B(x,y)$ the potential density and resolvent density for the corresponding operators G^B and G_a^B on B and with $G^B G_a^B(x,y)$ the density for the operator $G^B G_a^B$. Formulae are known for the Poisson kernel and potential density on spheres which imply that the last line in (4.4) is C^∞ on B. (See §§1.10 and 1.11 in Durrett (**1984**).) Thus h is C^∞ on B and therefore, after varying B, on D. Reversing (4.2) gives (4.1).   Q.E.D.

LEMMA 4.2. *For f in F^0 and for every x in D*

$$(4.5) \qquad \qquad \underset{t\uparrow\sigma}{\mathrm{Lim}}\, f(X_t) = 0 \quad a.e.\ dP_x.$$

PROOF. Certainly (4.5) is true for f in $C^\infty_{\mathrm{com}}(D)$. For general f in F^0 there exists by Reg2 in §1 a sequence f_k, $k \geq 1$, in $C^\infty_{\mathrm{com}}(D)$ such that $f_k \to f$ in F^0. By the argument on the bottom half of p. 4.4 in SMP, and possibly after selecting a subsequence from the f_k, we conclude that for almost every x in D and a.e. dP_x, $f_k(X_t) \to f(X_t)$ uniformly in $t \leq \sigma$, implying (4.5). The "almost every x" restriction is removed by observing that if Γ is the subset of Ω where (4.5) is true, then $h(x) = P_x(\Gamma)$ is a harmonic function which $= 1$ almost everywhere and therefore everywhere on D.   Q.E.D.

LEMMA 4.3. *Let $a > 0$ and let h be in H_a. Then for every x in D*

$$(4.6) \qquad \qquad h(X_{\sigma-}) = \underset{t\uparrow\sigma}{\mathrm{Lim}}\, h(X_t)$$

exists a.e. dP_x and

$$(4.7) \qquad \qquad h(x) = E_x e^{-a\sigma} h(X_{\sigma-}).$$

Moreover (4.6) is true with h replaced by general f in F^r and also

$$(4.8) \qquad \qquad H_a f(x) = E_x e^{-a\sigma} f(X_{\sigma-}).$$

PROOF. As in the proof of Lemma 4.1

$$(4.9) \qquad \qquad h(x) = E_x e^{-a\tau} h(X_\tau)$$

for D' an open set with closure contained in D, for x in D' and with τ denoting the exit time from D'. Also

$$E_x\{e^{-a\tau} h(X_\tau)\}^2 = E_x\left\{ h(X_0) + \int_0^\tau \nabla h(X_t) e^{-at} \cdot dX_t \right\}^2$$

$$= h^2(x) + E_x \int_0^\tau |\nabla h|^2(X_t) e^{-2at}\, dt$$

by Ito's formula (p. 64 in Durrett (**1984**)) which is applicable since h is in C^∞ by Lemma 4.1. The last expression is dominated independent of D' by $G^0|\nabla h|^2(x)$ which is finite for almost every x by Definition 5.1 in SMP. For such x the process $e^{-at} h(X_t)$, $t \geq 0$, is an L^2 bounded martingale and passage to the limit in D' gives (4.6) and (4.7) for almost every x. The "almost every" restriction is removed as in the proof of Lemma 4.2. The conclusions for general f in F^r follow from Lemma 4.2 since $f - H_a f$ belongs to F^0.   Q.E.D.

THEOREM 4.4 (DIRICHLET PRINCIPLE). *For f in F^r the function*

(4.10) $$H_0 f(x) = E_x f(X_{\sigma-})$$

belongs to F^r and satisfies

(4.11) $$E(H_0 f, H_0 f) \le E(f, f).$$

Also $H_0 f$ is harmonic, $f - H_0 f$ is in F^0, and

(4.12) $$E(H_0 f, g) = 0, \qquad g \text{ in } F^0.$$

PROOF. As above we can assume $f \ge 0$ without losing generality. For $0 < a < b$ define

$$Q_{a,b}(f, f) = (b - a) \int dx \, H_a f(x) H_b f(x).$$

The calculation

$$
\begin{aligned}
H_a f(x) - H_b f(x) &= E_x \{ e^{-a\sigma} - e^{-b\sigma} \} f(X_{\sigma-}) \\
&= (b - a) E_x \int_0^\sigma dt \, e^{-(b-a)t} e^{-a\sigma} f(X_{\sigma-}) \\
&= (b - a) E_x \int_0^\sigma dt \, e^{-bt} H_a f(X_t)
\end{aligned}
$$

establishes the identity

(4.13) $$H_a f - H_b f = (b - a) G_b^0 H_a f$$

which implies that $H_a f - H_b f$ is in F^0. (This can also be deduced from the already used fact that both $f - H_a f$ and $f - H_b f$ belong to F^0.) Thus

$$E_a(H_a f, H_a f - H_b f) = 0, \quad E_b(H_b f, H_a f - H_b f) = 0$$

and subtraction gives

$$E_a(H_a f - H_b f, H_a f - H_b f) = (b - a) \int dx \, H_b f(x) \{ H_a f(x) - H_b f(x) \}$$

which implies

(4.14) $$E_a(H_a f - H_b f, H_a f - H_b f) \le Q_{a,b}(f, f).$$

Also, again since $H_a f - H_b f$ is in F^0,

$$
\begin{aligned}
E_b(H_b f, H_b f) &- E_a(H_a f, H_a f) \\
&= E_b(H_a f, H_b f) - E_a(H_a f, H_b f) \\
&= (b - a) \int H_a f(x) H_b f(x) \, dx
\end{aligned}
$$

and therefore

(4.15) $$E_b(H_b f, H_b f) - E_a(H_a f, H_a f) = Q_{a,b}(f, f).$$

the inequality (4.14) implies $Q_{a,b}(f, f) \ge 0$ and then equality (4.15) implies that $E_a(H_a f, H_a f)$ decreases with a and therefore $Q_{a,b}(f, f) \to 0$ as $b \downarrow 0$. Finally (4.14) then implies that $H_a f$, $a > 0$, is E Cauchy as $a \downarrow 0$. The same is true for $(f - H_a f)_{a>0}$ which belong to F^0 and by Lemma 1.2 has a limit in F^0. Equivalently,

the $H_a f$ converge in F^r as $a \downarrow 0$ to something which in fact must be $H_0 f$ since $f \geq 0$ implies that $H_a f \to H_0 f$ pointwise by the monotone convergence theorem. We have shown that $H_0 f$ is in F^r and that $f - H_0 f$ is in F^0. The inequality (4.11) and the orthogonality relation (4.12) follow by passage to the limit $a \downarrow 0$. Finally harmonicity follows from (4.10).   Q.E.D.

NOTATION. H_0 denotes also the collection of harmonic functions in F^r.

For f in $C^\infty_{\text{com}}(D)$ and h in H_0, integration by parts gives

$$(4.16) \qquad\qquad E(f, h) = 0$$

valid for f in F^0 and h in H_0 by passage to the limit in f. We can view H_0 as the E orthogonal complement of F^0 in F^r, keeping in mind that E is not a true inner product on F^r since $E(1, 1) = 0$.

By harmonicity of $H_0 f$ the integral on the right in (4.10) converges absolutely for all f in F^r and for all x in D.

5. The boundary.

LEMMA 5.1. *There exists a countable collection Q_0 of bounded functions in H_0 which are E dense there.*

PROOF. If we knew that F^r were a separable Hilbert space then by contractivity there would be a dense sequence of bounded functions f_m, $m \geq 1$, in F^r. The images $H_0 f_m$ are E dense in H_0 and bounded by (4.10). So we need only prove that F^r is separable.

Let $E(\lambda)$, $\lambda \geq 0$, be the spectral projections associated to the nonnegative definite selfadjoint operator $-A^r$. (See Kunze and Segal (**1968**).) For integer $k \geq 1$ let $M_k = E(k)L^2$. Each M_k is a closed, possibly empty, subspace of both L^2 and F^r on which the L^2 and E_1 norms are equivalent. Separability of the M_k and therefore of F^r then follow from separability of L^2.   Q.E.D.

Represent Q_0 from Lemma 5.1 as a sequence $\{h_1, h_2, \dots\}$ and for $m \geq 1$ discard h_m if it is (0 for $m = 1$) linearly dependent on $h_1, \dots, h_{m-1}$. Denote by Q_1 the set of functions surviving this depletion procedure.

For bounded f, g in H_0 the product

$$(5.1) \qquad\qquad fg = \tfrac{1}{4}(f + g)^2 - \tfrac{1}{4}(f - g)^2$$

belongs to F^r by contractivity and therefore

$$(5.2) \qquad\qquad f \# g = H_0(fg)$$

is a bounded function in H_0.

Denote by Q the algebra over the rationals generated by Q_1, multiplication being the operator $\#$ of (5.2). Of course Q also is a countable E_1 dense subset of H_0.

Let W be the collection of functions w defined on f in Q so that $|w(f)| \leq \|f\|_\infty$ (usual L^∞ norm on D). Equip W with the usual product topology so that it becomes a separable compact Hausdorff space. The *boundary* $\partial^\# D$ is the subset of w in W satisfying

$$(5.3) \qquad\qquad w(f \# g) = w(f)w(g), \quad w(af) = aw(f),$$
$$w(f + g) = w(f) + w(g), \quad f \geq 0 \text{ implies } w(f) \geq 0$$

with f, g in Q and with a rational. Then $\partial^{\#} D$ is a closed subspace of W and is itself a separable compact Hausdorff space.

All functions f in Q are *extended* to $\partial^{\#} D$ by the formula $f(w) = w(f)$. Our *state space* then is $D^{\#} = D \cup \partial^{\#} D$. Equip $D^{\#}$ with the coarsest topology so that D is open and its subspace topology agrees with its original Euclidean topology and such that every f in Q is continuous on $D^{\#}$. Then for w in $\partial^{\#} D$, sets $\{x$ in $D^{\#}$: $|f_j(x) - f_j(w)| < \varepsilon_j$ for $1 \leq j \leq m\}$ with the f_j in Q form a neighborhood basis at w. Thus x_k, $k \geq 1$, in D converge to w in $\partial^{\#} D$ if and only if $f(x_k) \to f(w)$ for all f in Q.

LEMMA 5.2. *For every x in D and a.e. dP_x $X_\sigma = \mathrm{Lim}_{t \uparrow \sigma} X_t$ exists in the topology of $D^{\#}$ and belongs to $\partial^{\#} D$.*

PROOF. For f in Q the process $f(X_t)$, $0 \leq t < \sigma$, is a bounded martingale and so $L(f) = \mathrm{Lim}_{t \uparrow \sigma} f(X_t)$ exists a.e. dP_x for x in D. Since Q is countable, L is a random functional on Q defined a.e. dP_x, x in D. Certainly L satisfies the second, third and fourth identities in (5.3). For f, g in Q $fg = f \# g + h$ with h in F^0. By Lemma 4.2 $h(X_t) \to 0$ as $t \uparrow \sigma$ a.e. dP_x and the first identity in (5.3) is verified.   Q.E.D.

NOTATION. From now on X_σ and X_{σ^*} denote the appropriate limit in $\partial^{\#} D$ rather than the dead point ∂.

The next lemma follows from an isoperimetric result of Weinberger (**1962**). This and more is proved by Aizenman and Simon (**1982**), Appendix 3.

LEMMA 5.3. *There exists $C > 0$ such that*

$$(5.4) \qquad \int G^0(x, y)\, dy \leq C$$

for all x in D.

PROOF. Choose $R > 0$ such that x in D and $|x - y| \geq R$ implies that y is not in D. Then the integral is

$$\leq E_x \inf\{t > 0 \colon |B_t - x| = R\}$$
$$= E_0 \inf\{t > 0 \colon |B_t| = R\}.$$

For dimension $n \geq 3$ the last expression is $c \int_0^R dr\, r^{n-1} r^{2-n}$ and for $n = 2$, $c \int_0^R dr\, r \log(R/r)$.   Q.E.D.

DEFINITION. *Harmonic measure $d\nu$ is defined on Borel subsets V of $\partial^{\#} D$ by*

$$(5.5) \qquad \nu(V) = \int dx\, P_x(X_\sigma \text{ is in } V).$$

For h in Q and x in D

$$E_x h^2(X_\sigma) = E_x \left\{ h(X_0) + \int_0^\sigma \nabla h(x_t)\, dX_t \right\}^2$$
$$= h^2(x) + E_x \int_0^\sigma |\nabla h|^2(X_t)\, dt$$

and integration in x gives

$$\int_\Delta d\nu(z) h^2(z) = \int dx\, h^2(x) + \int dx \int dy\, G^0(x,y)|\nabla h|^2(y)$$
$$\leq \int dx\, h^2(x) + C \int dy |\nabla h|^2(y)$$

by Lemma 5.3. This establishes the useful estimate

(5.6)
$$\int d\nu(z) h^2(z) \leq c E_1(h,h)$$

for h in Q. Bounded h in H_0 can be E_1 approximated by functions in Q and then general h in H_0 can be E_1 approximated by its truncations. Thus (5.6) allows us to extend by continuity from h in Q to h in H_0 the definition of h on $\partial^\# D$ as a member of $L^2(\partial^\# D, d\nu)$. General f in F^r is defined on $\partial^\# D$ to agree with $H_0 f$. In particular f in F^0 vanishes identically on $\partial^\# D$. Of course (5.6) continues to be true for h in H_0.

CONVENTION. From now on

$$H_0(x,V) = P_x(X_\sigma \text{ is in } V),$$
$$H_a(x,V) = E_x e^{-a\sigma} I(X_\sigma \text{ is in } V).$$

If a subset V of $\partial^\# D$ is $d\nu$ null—that is $\nu(V) = 0$—then

(5.7)
$$\int dx\, H(x,V) = 0$$

and so $H_0(x,V) = 0$ for almost every and therefore every x since $H_0(x,V)$ is harmonic. Thus the measures $H_0(x,dz)$, x in D are all absolutely continuous with respect to harmonic measure ν and so for f in F^r the random variable $f(X_\sigma)$ is well defined a.e. dP_x, x in D. Also if $f_k \to f$ in F^r then for a subsequence $f_k(X_\sigma) \to f(X_\sigma)$ a.e. dP_x, x in D.

LEMMA 5.4. *For f in F^r, x in D and a.e. dP_x*

(5.8)
$$\operatorname*{Lim}_{t \uparrow \sigma} f(X_t) = f(X_\sigma).$$

PROOF. For f in F^0 the lemma is the same as Lemma 4.2 and so by Theorem 4.5 we need only consider the case when f is in H_0. The proof of Lemma 5.2 implies

(5.9)
$$f(x) = H_0 f(x) = E_x f(X_\sigma)$$

for f in Q. The extension to general f in H_0 follows by the same approximation used for (5.6). Then (5.8) follows from the martingale convergence theorem (p. 305 in Durrett (**1984**)). Q.E.D.

6. Normal derivative and boundary conditions.

LEMMA 6.1. *Let D' be an open set with closure contained in D, and let f be in F^0. Then*

(6.1)
$$E_\infty I(\sigma(D') < \infty) f^* < \infty$$

where $f^* = \sup_t |f(X_t)|$.

PROOF. Let $K = \{x \text{ in } D: |f(x)| > t\}$ for $t > 0$. By (3.0) in SMP, $\text{Cap}(K) \leq t^{-2}E(f,f)$. By (5.9) in SMP, $\text{Cap}(K) = P_\infty(\sigma(K) < \infty)$. Thus

$$(6.2) \qquad P_\infty(f^* > t) \leq t^{-2}E(f,f)$$

and so

$$E_\infty I(\sigma(D') < \infty)f^* \leq P_\infty(\sigma(D') < \infty) + E_\infty(f^* - 1)^+$$

$$= P_\infty(\sigma(D') < \infty) + \int_0^\infty P_\infty(\{f^* - 1\} > t)\, dt$$

$$= P_\infty(\sigma(D') < \infty) + E(f,f)\int (t+1)^{-2}\, dt. \quad \text{Q.E.D.}$$

LEMMA 6.2. *Let f be in the reflected space F^r and assume that Δf is in L^2. Then there exists a polar set N such that for x in the complement $D \backslash N$ and a.e. dP_x*

$$(6.3) \qquad f(X_{t\wedge\sigma}) = f(X_0) + \int_0^{t\wedge\sigma} \nabla f(X_s) \cdot dX_s$$

$$+ \frac{1}{2}\int_0^{t\wedge\sigma} \Delta f(X_s)\, ds$$

for all $t \geq 0$.

PROOF. Of course (6.3) is a standard formula in the "Ito calculus" when f is C^2. By Theorem 4.5 there exists (harmonic and therefore C^∞) h in H_0 such that $f - h$ is in F^0. Therefore we need only consider f in F^0. Extend f to R^n by setting $f(x) = 0$ for x in $R^n \backslash D$.

Let $\varphi(y) \geq 0$ be radial (depends only on $|y|$), C^∞, supported in the unit ball centered at the origin and such that $\int_{R^n} \varphi(y)\, dy = 1$. Assume also that $\varphi(|y|)$ is nonincreasing in $|y|$. For $k \geq 1$ let

$$(6.4) \qquad \varphi_k(y) = k^n \varphi(ky),$$

$$f_k(y) = f * \varphi_k(y) = \int_{R^n} dy\, \varphi_k(y) f(x - y).$$

Each f_k is C^∞ and so (6.3) is true with f replaced by f_k.

Certainly $f_k \to f$ in L^2 on R^n and by the description of F^0 given in the Introduction ∇f is in L^2 on R^n and therefore also $\nabla f_k \to \nabla f$ in L^2 on R^n. (Unfortunately it is not true in general that $\Delta f_k \to \Delta f$ in L^2 on R^n and this limits applicability of the argument below.) For dimension $n \geq 3$ Brownian motion on R^n is transient and so Theorem 3.13 in SMP guarantees the existence of a subsequence of the f_k which converges to f pointwise modulo a polar set. (It is easy to see that a subset of D is polar for Brownian motion on R^n if and only it is for the absorbed process on D.) For $n = 2$ the analogous argument works with R^n replaced by a sufficiently large disk, obtaining a transient process. Thus in every case there exists a polar set N' of D and a subsequence $f_k^\wedge$, $k \geq 1$, of f_k, $k \geq 1$, such that

$$(6.5) \qquad f_k^\wedge(x) \to f(x), \qquad x \text{ in } D\backslash N'.$$

In completing the proof we use the elementary fact that for $g \geq 0$ in L^1 convolutions $\varphi_k * g$ are uniformly pointwise dominated by the Hardy-Littlewood maximal function Mg, treated, for example, in Stein (**1970a**). This domination follows from

$$\varphi_k * g(x) = \int_{R^n} dy\, \varphi_k(y) g(x - y)$$

$$= \int_0^\infty dr\, r^{n-1} \varphi_k(r) \int_S d\sigma(z) g(x - rz)$$

(S being the unit sphere in R^n and $d\sigma$ uniform measure (surface measure) on S)

$$= \int_0^\infty dr(-\varphi_k'(r)) \int_0^r ds\, s^{n-1} \int_S d\sigma\,(z) g(x - sz)$$

$$\leq c \int_0^\infty dr(-\varphi_k'(r)) r^n Mg(x)$$

$$= c Mg(x)$$

with usual convention that unspecified constant c may vary from line to line.

Also we need D_l, $l \geq 1$, an increasing sequence of open sets with closures contained in D and with union D.

For a given l and for k sufficiently large and for x in D_l the convolution $f_k^\wedge$ depends only on the values of f in some compact subset of D and so $\Delta f_k^\wedge(x)$ is not effected by contributions to Δf on the boundary of D. So by the inequalities for Mg proved in §I.1 of Stein (**1970a**), the functions

$$|\nabla f|^* = \sup_{k \geq k_0} |\nabla f_k^\wedge|, \quad |\Delta f|^* = \sup_{k \geq k_0} |\Delta f_k^\wedge|$$

(k_0 depending on D_l) are in L^2 on D_l. By §1 in SMP

(6.6) $$S(x) = E_x \int_0^{\sigma(D_l)} dt\, \{|\nabla f|^{*2}(X_t) + |\Delta f|^*(X_t)\} < \infty$$

for almost every x in D_l. But $S(x)$ is superharmonic in D_l and therefore $S(X_{t \wedge \sigma(D_l)})$, $t \geq 0$, is a supermartingale. This implies that if

$$N_l = \{x \text{ in } D_l : S(x) = +\infty\}$$

then $P_x(\sigma(N_l) < \infty) = 0$ for x in D_l such that $S(x) < \infty$. Therefore N_l is polar.

By the corollary at the bottom of p. 5 in Stein (**1970a**)

(6.7) $$\nabla f_k^\wedge(x) \to \nabla f(x), \quad \Delta f_k^\wedge(x) \to \Delta f(x)$$

as $k \to \infty$ for almost every x in D_l. (This also depends on domination by the Hardy-Littlewood maximal function established above.) If Borel K is dx null then for every x in D

$$E_x \int_0^\sigma I(X_s \text{ is in } K)\, ds = 0.$$

Thus (6.7) implies that for x in D_l and a.e. dP_x

(6.8) $$\nabla f_k^\wedge(X_s) \to \nabla f(X_s), \quad \Delta f_k^\wedge(X_s) \to \Delta f(X_s)$$

for almost every s such that $0 \leq s < \sigma(D_l)$. This combines with (6.6) to imply that for x in $D_l \backslash N_l$

$$(6.9) \qquad \int_0^{t \wedge \sigma(D_l)} \Delta f_k^\wedge(X_s) \, ds \rightarrow \int_0^{t \wedge \sigma(D_l)} \Delta f(X_s) \, ds$$

a.e. dP_x and that

$$(6.10) \qquad E_x \int_0^{t \wedge \sigma(D_l)} |\nabla f_k^\wedge(X_s) - \nabla f(X_s)|^2 \, ds \rightarrow 0.$$

By §2.1 and Appendix A.5 in Durrett (**1984**), (6.10) implies

$$(6.11) \qquad \int_0^{t \wedge \sigma(D_l)} \nabla f_k^\wedge(X_s) \cdot dX_s \rightarrow \int_0^{t \wedge \sigma(D_l)} \nabla f(X_s) \cdot dX_s$$

almost everywhere dP_x. Now (6.5), (6.9) and (6.11) imply (6.3) with σ replaced by $\sigma(D_l)$ for x not in the polar set $N' \cup N_l$.

Finally

$$(6.12) \qquad E_x \int_0^\sigma ds \, \{ \Delta f(X_s) + |\nabla f|^2 (X_s) \} < \infty$$

a.e. dP_x with x in $D \backslash N''$ with N'' polar by the same argument which gave (6.6) modulo a polar set. This allows us to pass to the limit $l \uparrow \infty$ in (6.3) with σ replaced by $\sigma(D_l)$ to obtain (6.3) as written a.e. dP_x for x in $D \backslash N$ with N the polar set $N' \cup N'' \cup (\bigcup_{l=1}^\infty N_l)$. Q.E.D.

LEMMA 6.3. *Let h be in H_0. Then*
1. $\langle h \rangle^* = \sup_{\sigma^* \leq s, t \leq \sigma} |h(X_t) - h(X_s)|$ *is in $L^2(\Omega_\infty, dP_\infty)$.*
2. $E_\infty \{ h(X_\sigma) - h(X_{\sigma^*}) \}^2 = \int dx |\nabla h|^2(x)$.
3. *For f in F^r with Δf in L^2*

$$\int dx \, \nabla h(x) \cdot \nabla f(x) = E_\infty \{ h(X_\sigma) - h(X_{\sigma^*}) \} \{ f(X_\sigma) - f(X_{\sigma^*}) \}.$$

PROOF. Let D' be open with closure contained in D. Then

$$E_\infty \sup_{t \geq \sigma(D')} |h(X_t) - h(X_{\sigma(D')})|^2 I(\sigma(D') < \infty)$$

$$\leq 4 E_\infty |h(X_\sigma) - h(X_{\sigma(D')})|^2 I(\sigma(D') < \infty)$$

(by Doob's maximal inequality for $p = 2$, Appendix A.6 in Durrett (**1984**))

$$= 4 E_\infty I(\sigma(D') < \infty) \int_{\sigma(D')}^\sigma dt \, |\nabla h|^2(X_t)$$

(§2.1, ibid.)

$$\leq 4 \int dx \, |\nabla h|^2(x)$$

by (2.3). By time reversal invariance, conclusion 3 in Theorem 2.1, the same is true with $\sigma(D')$ replaced by the last exit time from D', $\sigma^*(D')$ and with $t \geq \sigma(D')$ replaced by $t \leq \sigma^*(D')$ in the interval of integration. Conclusions 1 and 2 follow from these estimates.

For conclusion 3 note first that Lemma 6.2 implies

$$(6.13) \quad \frac{1}{2} I(\sigma(D)' < \infty) \int_{\sigma(D')}^{\sigma} \Delta f(X_t)\, dt$$

$$= I(\sigma(D') < \infty) \left\{ f(X_\sigma) - f(X_{\sigma(D')}) - \int_{\sigma(D')}^{\sigma} \nabla f(X_t) \cdot dX_t \right\}$$

a.e. dP_∞. By Lemma 5.4 and Theorem 2.1

$$(6.14) \qquad\qquad I(\sigma(D') < \infty) f(X_{\sigma(D')}) \to f(X_{\sigma^*})$$

a.e. dP_∞ as D' increases to D. Also

$$(6.15) \qquad I(\sigma(D') < \infty) \int_{\sigma(D')}^{\sigma} \Delta f(X_t)\, dt \to \int_{\sigma^*}^{\sigma} \Delta f(X_t)\, dt$$

a.e. dP_∞ since

$$E_x \int_{\sigma^*}^{\sigma} |\Delta f|(X_t)\, dt < \infty.$$

Also

$$I(\sigma(D') < \infty) \int_{\sigma(D')}^{\sigma} \nabla f(X_t) \cdot dX_t$$

converges in $L^2(\Omega_\infty, dP_\infty)$ since

$$E_\infty \int_{\sigma^*}^{\sigma} |\nabla f|^2(X_t)\, dt = \int dx\, |\nabla f|^2(x) < \infty.$$

This together with (6.14) and (6.15) allows us to pass to the limit $D' \uparrow D$ in (6.13) and obtain

$$(6.16) \qquad \frac{1}{2} \int_{\sigma^*}^{\sigma} \Delta f(X_t)\, dt = f(X_\sigma) - f(X_{\sigma^*}) - \int_{\sigma^*}^{\sigma} \nabla f(X_t) \cdot dX_t.$$

This implies in particular that the left-hand side is in $L^2(\Omega_\infty, dP_\infty)$ since the terms on the right are. Conclusion 3 follows since by time reversal invariance

$$E_\infty \left(\int_{\sigma^*}^{\sigma} \Delta f(X_t)\, dt \right) \left(\int_{\sigma^*}^{\sigma} \nabla h(X_t) \cdot dX_t \right)$$

$$= 2E_\infty \int_{\sigma^*}^{\sigma} \Delta f(X_t) \int_t^{\sigma} \nabla h(X_s) \cdot dX_s = 0. \quad \text{Q.E.D.}$$

As noted in the introduction f belongs to domain(A^r) with $A^r f = g$ if and only if the following two conditions are satisfied.

GEN1. f is in F^r.

GEN2. For all h in F^r

$$(6.18) \qquad\qquad E(f, h) + \int dx\, g(x) h(x) = 0.$$

For h in $C^\infty_{\text{com}}(D)$ integration by parts gives $g = \frac{1}{2}\Delta f$. Therefore Gen2 can be replaced by the following two conditions.

GEN2.1. Δf is in L^2.

GEN2.2. For all h in F^r

$$(6.19) \qquad \int dx\, \nabla f(x) \cdot \nabla h(x) + \int dx\, \Delta f(x) h(x) = 0.$$

If f satisfies conditions Gen1 and Gen2.1 then Gen2.2 is equivalent to (6.19) for bounded h in F^r. By Theorem 4.5 there exists bounded h_1 in F^0 such that $h - h_1$ is in H_0. We have seen that (6.19) is true for h_1 and so we need only verify (6.19) for bounded h in H_0.

DEFINITION. φ defined on $\partial^\# D$ is a *test function* if it is the boundary value (restriction to $\partial^\# D$) of some f in F^r.

DEFINITION. For f in F^r its *normal derivative* is the function $\partial f/\partial n$ defined on test functions φ by

$$\partial f/\partial n(\varphi) = \lim_{k\uparrow\infty} E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty)\{f(X_{\sigma^*}) - f(X_{\sigma(D_k)})\}.$$

The D_k, $k \geq 1$ are as in the proof of Lemma 6.2.

Now we prove the theorem stated in the Introduction. From (6.3) follows

$$E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty)\{f(X_{\sigma^*}) - f(X_{\sigma(D_k)})\}$$
$$= E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty)\{f(X_{\sigma^*}) - f(X_\sigma)\}$$
$$+ E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \int_{\sigma(D_k)}^\sigma \nabla f(X_t) \cdot dX_t$$
$$+ E_\infty \varphi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \int_{\sigma(D_k)}^\sigma \tfrac{1}{2}\Delta f(X_t)\, dt.$$

By time reversal invariance and Lemma 6.3 the first term converges to

$$= \tfrac{1}{2} E_\infty \{\varphi(X_\sigma) - \varphi(X_{\sigma^*})\}\{f(X_\sigma) - f(X_{\sigma^*})\}$$
$$= \tfrac{1}{2} \int dx\, \nabla H_0 \varphi(x) \cdot \nabla f(x).$$

The second term vanishes by conclusion 1 in Theorem 2.1. Passage to the limit in the third term gives

$$E_\infty \varphi(X_{\sigma^*}) \int_{\sigma^*}^\sigma \tfrac{1}{2}\Delta f(X_t)\, dt$$

which by time reversal invariance

$$= E_\infty \int_{\sigma^*}^\sigma \tfrac{1}{2}\Delta f(X_t)\, dt \varphi(X_\sigma).$$

Replacing $\varphi(X_\sigma)$ by its conditional expectation $H_0\varphi(X_t)$ and using (2.3), we get

$$\frac{1}{2} \int dx\, H_0\varphi(x)\Delta f(x).$$

Thus

(6.22) $$\partial f/\partial n(\varphi) = \frac{1}{2} \int dx\, \{\nabla f(x) \cdot \nabla H_0\varphi(x) + \Delta f(x) H_0\varphi(x)\}$$

and we are done since every h in H_0 can be represented $h = H_0\varphi$.   Q.E.D.

7. Direct reduction to the classical formula for D smooth. In this section D is bounded and has C^∞ boundary. We begin by showing that the abstract boundary $\partial^\# D$ of §5 can be constructed so that it is identical to the Euclidean boundary ∂D.

In Lemma 5.1 choose Q_0 so that each f in Q_0 is C^∞ on the closure of D and so that the corresponding algebra Q, restricted to the boundary ∂D, is dense in

$C(\partial D)$. Every point w in $\partial^{\#} D$ is a bounded linear functional on Q which is nonnegative on nonnegative functions. So by the Riesz theorem there is a unique Radon measure $d\nu$ on ∂D such that

$$w(f) = \int_{\partial D} d\nu(z) f(z), \quad f \text{ in } Q.$$

The multiplicative property $w(f \# g) = w(f) w(g)$ in (5.3) implies that $d\nu$ has mass one and is concentrated at a point ψw in ∂D. This map $\psi: \partial^{\#} D \to \partial D$ is easily seen to be bijective. Finally, it is easy to see that the Euclidean topology on the closure $D^{-} = D \cup \partial D$ agrees with the topology specified in §5. This completes the identification of $\partial^{\#} D$ with ∂D.

Since D is C^{∞} the *exterior unit normal* N_z is well defined and a smooth function of z in ∂D. For f defined on the closure D^{-} its "classical" normal derivative is $D_n f(z) = \nabla f(z) \cdot N_z$.

The potential density $G^0(x, y)$ is C^{∞} for $x \neq y$ in D^{-}. (See the subsection on the "method of layer potentials" in Jerison and Kenig (**1982**).) In particular, for x in D

$$(7.1) \qquad D_n G^0(x, y) = \nabla_z G^0(x, z) \cdot N_z$$

is well defined and continuous for z in ∂D.

Let f be continuous on D^{-} and harmonic in D. For x in D

$$-\tfrac{1}{2} \Delta_y G^0(x, y) = \delta_x(dy)$$

the unit point mass at x, and so

$$(7.2) \qquad \begin{aligned} f(x) &= -\frac{1}{2} \int dy \, f(y) \Delta_y G^0(x, y) \\ &= \frac{1}{2} \int dy \{ G^0(x, y) \Delta f(y) - f(y) \Delta_y G^0(x, y) \} \\ &= \frac{1}{2} \int dy \, \mathrm{div} \{ G^0(x, y) \nabla f(y) - f(y) \nabla_y G^0(x, y) \} \\ &= -\frac{1}{2} \int_{\partial D} d\sigma(dz) f(z) D_n G^0(x, z) \end{aligned}$$

by the classical divergence theorem since $G^0(x, z) = 0$ for z in ∂D. Here $d\sigma$ denotes surface measure which can be defined in the usual way since D is smooth.

Here and below the operator D_n in the expression $D_n G^0(x, z)$ acts on the second variable.

Let $H_0(x, dy)$ denote the exit measure on ∂D with starting point x. That is, for $\phi \geq 0$ on ∂D

$$\int_{\partial D} H_0(x, dz) \phi(z) = E_x \phi(X_\sigma).$$

In particular for f continuous on D^{-} and harmonic on D

$$f(x) = \int H_0(x, dz) f(z).$$

Together with (7.2), this implies

$$(7.3) \qquad H_0(x, dz) = -\tfrac{1}{2} D_n G^0(x, z) \, d\sigma(z).$$

Assume that the origin 0 is in D. In this section it is convenient to take

(7.4) $$H(dz) = H_0(0, dz)$$

as "harmonic measure". Then

$$H(dz) = -\tfrac{1}{2} D_n G^0(0, z)\, d\sigma(z).$$

By Hunt and Wheeden (**1970**) the Martin kernel $K(x, y)$, which is continuous and strictly positive for x in D and y in D^-, satisfies $H_0(x, dz) = K(x, z)\, dH(z)$ and so for x in D and z in ∂D

(7.5) $$K(x, z) = D_n G^0(x, z)/D_n G^0(0, z)$$

except possibly for z sets of harmonic measure 0. By continuity, (7.5) is true whenever $D_n G^0(0, z) \neq 0$. We verify now that this is true for all z, and indeed that

(7.6) $$D_n G^0(x, z) < 0, \quad x \text{ in } D, \ z \text{ in } \partial D.$$

From (7.5) it follows that if $D_n G^0(0, z) = 0$ for a given z in ∂D, then $D_n G^0(x, z) = 0$ for all x in D. But this is impossible. Using local coordinates it is easy to construct smooth f vanishing on ∂D such that $D_n f(z) \neq 0$. Such f belongs to the domain of the absorbed generator A_0 and therefore

$$f(x) = -\frac{1}{2} \int G^0(x, y) \Delta f(y).$$

This implies

$$D_n f(z) = -\frac{1}{2} \int dy\, \Delta f(y) D_n G^0(y, z) = 0,$$

a contradiction which proves (7.6).

Let D_1 be open with closure contained in D and denote by h its equilibrium potential, that is, $h(x) = P_x(\sigma(D_1) < \infty)$. Then $h(z) = \int d\mu(y) G^0(y, z)$ with $d\mu$ equilibrium measure for D_1. For z in ∂D

$$D_n h(z) = \int d\mu(y) D_n G^0(y, z)$$

and since $d\mu$ is bounded, the above arguments imply that $D_n h$ is continuous and bounded away from zero on ∂D. Also ∇h is continuous on D^- since D is smooth.

Consider now f and ϕ as in §6---only assume in addition that Δf is bounded and ∇f continuous on D^-. We use now the idea of Gundy referred to in the introduction.

It is convenient to specify

$$D_k = \{x \text{ in } D \colon h(x) > 1/k\}.$$

Then

(7.7)
$$E_\infty \phi(X_{\sigma^\cdot}) I(\sigma(D_k) < \infty)\{f(X_{\sigma^\cdot}) - f(X(\sigma(D_k)))\}$$
$$= E_\infty \phi(X_{\sigma^\cdot}) I(\sigma(D_k) < \infty) E_\infty(\{f(X_{\sigma^\cdot}) - f(X(\sigma(D_k)))\} | X_{\sigma^\cdot}, \sigma(D_k) < \infty),$$

the last occurrence of E_∞ denoting conditional expectation given $X_{\sigma^\cdot}$ and the event $\sigma(D_k) < \infty$.

The thus conditioned process X_t, $\sigma^* \leq t \leq \sigma(D_k)$, is absorbed Brownian motion on the complement $D \backslash D_k$, conditioned on exiting via ∂D_k rather than ∂D. Since

$$kh(x) = P_x(\sigma(D_k) < \infty), \quad x \text{ in } D \backslash D_k,$$

the conditioned generator is

$$\frac{1}{2h}\Delta h = \frac{1}{2}\Delta + \frac{1}{h}\nabla h \cdot \nabla$$

and we can replace the conditional expectation on the right-hand side in (7.7) by

$$-\int_{\sigma^*}^{\sigma(D_k)} \left\{ \frac{1}{2}\Delta f(X_t) + \frac{1}{h}\nabla h \cdot \nabla f(X_t) \right\} dt$$

to get

$$-E_\infty \phi(X_{\sigma^*}) I(\sigma(D_k) < \infty) \int_{\sigma^*}^{\sigma(D_k)} \left\{ \frac{1}{2}\Delta f(X_t) + \frac{1}{h}\nabla h \cdot \nabla f(X_t) \right\} dt$$

and then

(7.8) $$-E_\infty \phi(X_{\sigma^*}) \int_{\sigma^*}^{\sigma(D_k)} \left\{ \frac{kh}{2}\Delta f(X_t) + \nabla kh \cdot \nabla f(X_t) \right\} dt$$

since $I(\sigma(D_k) < \infty)$ can be replaced by $kh(t)$ inside the integral by (2.1). The same argument can be made with f replaced by h to give

(7.9) $$E_\infty \phi(X_{\sigma^*}) I(\sigma(D_k) < \infty)\{h(X_{\sigma^*}) - h(X(\sigma(D_k)))\}$$
$$= E_\infty \phi(X_{\sigma^*}) \int_{\sigma^*}^{\sigma(D_k)} k|\nabla h|^2(X_t)\, dt.$$

With the help of the additive functional determined by $d\mu$ in SMP, the proof in §6 can be adapted to show that $\partial h / \partial n(\phi)$ exists and is equal to

(7.10) $$\frac{1}{2}\int \nabla h \cdot \nabla H_0\phi(x)\, dx - \int d\mu(x) H_0\phi(x) = -\int d\mu(x) H_0\phi(x)$$

since the dx integral vanishes by (4.12) because h is in F^0. From the definition of $\partial h / \partial n(\phi)$ it follows that (7.9) converges to (7.10) and in particular

(7.11) $$\lim_{k \uparrow \infty} E_\infty \phi(X_{\sigma^*}) \int_{\sigma^*}^{\sigma(D_k)} |\nabla h|^2(X_t)\, dt$$
$$= -\int d\mu(x) H_0\phi(x) = \int_{\partial D} d\sigma(z)\phi(z) D_n h(z),$$

the last step following from (7.3) integrated against $d\mu(x)$. Since ∇h is bounded away from 0 in any neighborhood of ∂D, the expression

$$\frac{h}{2}\Delta f(X_t) + \nabla h \cdot \nabla f(X_t)$$

in (7.8) is asymptotic as $t \downarrow \sigma^*$ to

$$(1/|\nabla h|^2)(X_{\sigma^*})\nabla f \cdot \nabla h(X_{\sigma^*})|\nabla h|^2(X_t)$$

and it follows that

(7.12) $$\partial f / \partial n(\phi) = \int_{\partial D} d\sigma(z)(1/|\nabla h|^2)\nabla h \cdot \nabla f(z)\phi(z) D_n h(z).$$

Since h vanishes on ∂D $\nabla h = -D_n h$, $\nabla h \cdot \nabla f = D_n h D_n f$ on ∂D. Substituting this into (7.12) and applying (7.11), we get

$$(7.13) \qquad \partial f / \partial n(\phi) = \int_{\partial D} \phi(z) D_n f(z) \, d\sigma(z),$$

the desired result.

We do not believe that (7.13) is valid for general f satisfying the hypotheses of our main theorem but certainly weaker additional assumptions would suffice. However questions about the domain D seem more promising. We know that (7.13) can be extended to Lipschitz domains using the techniques in Jerison and Kenig (**1982**)—especially Lemmas 3.3 and 3.4. Is (7.13) true whenever $d\sigma$ is well defined? Is there a version of (7.13) which is true more generally?

8. Two examples done another way.

EXAMPLE 1. The dimension $n = 1$ and $D = (0,1)$, the unit interval. The potential density is

$$G^0(x, y) = 2x \cap y(1 - x \cup y).$$

The approximating sets are

$$D_\varepsilon = (\varepsilon, 1 - \varepsilon) \quad \text{with } 0 < \varepsilon < 1/2.$$

The equilibrium measure μ_ε is

$$\mu_\varepsilon = (1/2\varepsilon)\{\delta_\varepsilon + \delta_{1+\varepsilon}\}$$

with δ_x denoting the usual point mass at x.

Suppose that f is in F^r with $\Delta f = f''$ in L^2. Then $\nabla f = f'$ is continuous on the closure of D. The points ε, $1 - \varepsilon$ can be reached only from 0, 1 respectively at the first entrance time $\sigma(D_\varepsilon)$. Thus for φ a test function

$$\begin{aligned}
E_\infty \varphi(X_{\sigma \cdot})\{f(X_{\sigma \cdot}) - f(X_{\sigma(D_\varepsilon)})\} \\
= (1/2\varepsilon)(\varphi(0)\{f(0) - f(\varepsilon)\} + \varphi(1)\{f(1) - f(1 - \varepsilon)\}) \\
\to -\tfrac{1}{2}\varphi f'(0) + \tfrac{1}{2}\varphi f'(1)
\end{aligned}$$

in agreement with (0.10).

EXAMPLE 2. The dimension $n = 2$ and D is the upper half plane $R_+^2 = \{(x, y) : x \text{ is real and } y > 0\}$. The approximating sets are shifted versions $D_\varepsilon = D + (0, \varepsilon)$. It is convenient to write here $B_\varepsilon = D \backslash D_\varepsilon$.

The main step is to find the Poisson kernel $K_\varepsilon(z, t)$ for B_ε conditioned on hitting the upper boundary $y = \varepsilon$ before the lower one $y = 0$. Here $z = (x, y)$ in D_ε and t labels (t, ε) in the upper boundary. Using the complex structure $(z = x + iy)$ on R^2, we define the map $w = e^{\pi z / \varepsilon}$ which takes B_ε into R_+^2 in such a way that the upper boundary of B_ε corresponds to the negative half axis in the boundary of R_+^2. Let $z = (x, y)$ in B_ε correspond to $w = (u, v)$ in R_+^2 and let t labeling a point in the upper boundary of B_ε correspond to $s < 0$. Then

$$u = e^{\pi x / \varepsilon} \cos(\pi y / \varepsilon), \quad v = e^{\pi x / \varepsilon} \sin(\pi y / \varepsilon), \quad s = -e^{\pi t / \varepsilon}.$$

Let h be bounded and harmonic on B_ε and continuous on the closure. Then h is harmonic also as a function of w and so by the theory of harmonic functions on R_+^2

(Stein (**1970a**)),

$$h(z) = h((\varepsilon/\pi)\log w)$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} ds \frac{h((\varepsilon/\pi)\log(-s))v}{(s-u)^2 + v^2}$$

$$= \frac{1}{\pi}e^{\pi x/\varepsilon}\sin\left(\frac{\pi y}{\varepsilon}\right)\int_{-\infty}^{0} ds \frac{h((\varepsilon/\pi)\log(-s))}{\{(s - e^{\pi x/e}\cos(\pi y/e))^2 + e^{2\pi x/\varepsilon}\sin^2(\pi y/\varepsilon)\}}$$

$$= \frac{1}{\pi}\sin\left(\frac{\pi y}{\varepsilon}\right)\int_{-\infty}^{0} ds \frac{h((\varepsilon/\pi)\log(-s))e^{\pi x/\varepsilon}}{\{e^{2\pi x/\varepsilon} + s^2 - 2se^{\pi x/\varepsilon}\cos(\pi y/\varepsilon)\}}.$$

Substituting $t = (s/\pi)\log(-s)$ for s and multiplying numerator and denominator by $e^{-2\pi t/\varepsilon}$ gives

$$h(z) = \left(\frac{1}{\varepsilon}\right)\sin\left(\frac{\pi y}{\varepsilon}\right)\int_{-\infty}^{+\infty} dt \frac{e^{\pi(x-t)/\varepsilon}}{\{e^{2\pi(x-t)/\varepsilon} + 1 + 2e^{\pi(x-t)/\varepsilon}\cos(\pi y/\varepsilon)\}}.$$

Finally, division by the exponential in the numerator gives

$$K_\varepsilon(z,t) = (1/\varepsilon)\sin(\pi y/\varepsilon)/\{e^{\pi(x-t)/\varepsilon} + e^{-\pi(x-t)/\varepsilon} + 2\cos(\pi y/\varepsilon)\}.$$

Equilibrium measure for D_ε is $(1/2\varepsilon)\times$ (Lebesgue measure) on the line $y = \varepsilon$. Extend φ to R_+^2 so that $\varphi(x,y) = \varphi(0,y)$. Then for $0 < a < \varepsilon$

$$E_\infty I(\sigma(D_\varepsilon) < \infty)\varphi(X_{\sigma(D_a)})\{f(X_{\sigma(D_a)}) - f(X_{\sigma(D_\varepsilon)})\}$$

$$= \frac{1}{2a\varepsilon}\sin\left(\frac{\pi a}{\varepsilon}\right)\int_{-\infty}^{+\infty} dt\,\varphi(t)\int_{-\infty}^{+\infty} dx \frac{\{f(t,a) - f(x,\varepsilon)\}}{\{e^{\pi(x-t)/\varepsilon} + e^{-\pi(x-t)/\varepsilon} + 2\cos(\pi a/\varepsilon)\}}.$$

Assume now that φ is bounded and continuous with compact support and that f is bounded and C^1 on the closure of R_+^2 with ∇f bounded. Then passage to the limit $a \downarrow 0$ gives

$$E_\infty\varphi(X_{\sigma\cdot})I(\sigma(D_\varepsilon) < \infty)\{f(X_{\sigma\cdot}) - f(X_{\sigma(D_\varepsilon)})\}$$

$$= \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} ds \left(\frac{1}{2\varepsilon}\right)\varphi(t)\{f(t,0) - f(s,\varepsilon)\}Q_\varepsilon(t-s)$$

with $Q_\varepsilon(t) = (1/\varepsilon)Q(t/\varepsilon)$ and $Q(t) = (\pi/4)\mathrm{sech}^2(\pi t/2)$. Replacing $(t - s)/\varepsilon$ by s in the ds integral, we get

$$\frac{1}{2\varepsilon}\int dt\,\varphi(t)\int ds\,Q(s)\{f(t,0) - f(t + s\varepsilon,\varepsilon)\}.$$

Finally

$$f(t,0) - f(t + \varepsilon s,\varepsilon) = \varepsilon s\partial f/\partial x(t,0) - \varepsilon\partial f/\partial y(t,0) + O(\varepsilon^2)$$

and since $\int Q(s)s\,ds = 0$ and $\int Q(s)ds = 1$, passage to the limit $\varepsilon \downarrow 0$ yields

$$\frac{\partial f}{\partial n(\varphi)} = -\frac{1}{2}\int dt\,\varphi(t)\frac{\partial f}{\partial y}(t,0)$$

in agreement with (0.10).

REFERENCES

1. R. Durrett (1984), *Brownian motion and martingales in analysis*, Wadsworth, Belmont, Calif.
2. M. Fukushima (1971), *On transition probabilities of symmetric strong Markov processes*, J. Math. Kyoto Univ. **12**, 431–450.
3. S. Port and C. Stone (1978), *Brownian motion and classical potential theory*, Academic Press, New York.
4. I. E. Segal and R. A. Kunze (1968), *Integrals and operators*, McGraw-Hill, New York.
5. M. L. Silverstein (1974), *Symmetric Markov processes*, Lecture Notes in Math., vol. 426, Springer-Verlag.
6. ____ (1976), *Boundary theory for symmetric Markov processes*, Lecture Notes in Math., vol. 516, Springer-Verlag.
7. E. M. Stein (1970a), *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J.
8. ____ (1970b), *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. of Math. Studies, no. 63, Princeton Univ. Press, Princeton, N.J.
9. G. A. Hunt (1960), *Markov chains and Martin boundaries*, Illinois J. Math. **4**, 313–340.
10. M. Weil (1970), *Quasi-processus*, Séminaire de Probabilitiés IV, Lecture Notes in Math., vol. 124, Springer-Verlag.
11. M. Fukushima (1969), *On boundary conditions for multidimensional Brownian motion with symmetric resolvents*, J. Math. Soc. Japan **21**, 485–526.
12. ____ (1967), *Construction of reflecting barrier Brownian motion for bounded domains*, Osaka J. Math. **4**, 183–215.
13. ____ (1980), *Dirichlet forms and Markov processes*, Kodansha; North-Holland.
14. R. F. Gundy (1987), Private communication.
15. K. Burdzy (1985), *Brownian excursions from hyperplanes and smooth surfaces*, Trans. Amer. Math. Soc. **295** (1985), 35–57.
16. W. Feller (1958), *On the intrinsic form for second order differential operator*, Illinois J. Math. **2**, 1–18.
17. D. Jerison and C. Kenig (1982), *Boundary value problems on Lipschitz domains*, Math. Assoc. Amer. Stud., vol. 23, pp. 1–68.
18. R. Hunt and R. Wheeden (1970), *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **147**, 505–527.
19. M. Aizenman and B. Simon (1982), *Brownian motion and Harnack's inequality for Schrodinger operators*, Comm. Pure Appl. Math. **35**, 209–271.
20. H. F. Weinberger (1962), *Symmetrization in uniformly elliptic problems*, Studies in Mathematical Analysis and Related Topics: Essays in honor of Pólya, Stanford Univ. Press, Stanford, Calif., pp. 424–428.

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING, CHINA (Current address of Gong Guanglu and Qian Minping)

DEPARTMENT OF MATHEMATICS, DEPARTMENT OF MATHEMATICS, BOX 1146, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130 (Current address of M. L. Silverstein)